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A non-perturbative study of 4d U(1) non-commutative gauge theory — the fate of one-loop instability

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Abstract: Recent perturbative studies show that in 4d non-commutative spaces, the trivial (classically stable) vacuum of gauge theories becomes unstable at the quantum level, unless one introduces sufficiently many fermionic degrees of freedom. This is due to a negative IR-singular term in the one-loop effective potential, which appears as a result of the UV/IR mixing. We study such a system non-perturbatively in the case of pure U(1) gauge theory in four dimensions, where two directions are non-commutative. Monte Carlo simulations are performed after mapping the regularized theory onto a U(N) lattice gauge theory in $d = 2$. At intermediate coupling strength, we find a phase in which open Wilson lines acquire non-zero vacuum expectation values, which implies the spontaneous breakdown of translational invariance. In this phase, various physical quantities obey clear scaling behaviors in the continuum limit with a fixed non-commutativity parameter θ, which provides evidence for a possible continuum theory. The extent of the dynamically generated space in the non-commutative directions becomes finite in the above limit, and its dependence on θ is evaluated explicitly. We also study the dispersion relation. In the weak coupling symmetric phase, it involves a negative IR-singular term, which is responsible for the observed phase transition. In the broken phase, it reveals the existence of the Nambu-Goldstone mode associated with the spontaneous symmetry breaking.

Keywords: Matrix Models, Non-Commutative Geometry.

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1. Introduction

Non-commutative (NC) geometry [1, 2] has been studied extensively as a modification of our notion of space-time at short distances, possibly due to effects of quantum gravity [3]. It has recently attracted much attention since gauge theories on a NC geometry have been shown to appear as a low energy limit of string theories with a background tensor field [4]. At the classical level, introducing non-commutativity to the space-time coordinates modifies the ultraviolet dynamics of field theories (the interaction becomes non-local at the scale of non-commutativity), but not the infrared properties. The latter changes at the quantum level, however, due to the so-called UV/IR mixing effect [5]. As a consequence, one cannot retrieve the corresponding commutative theory by sending the non-commutativity parameter to zero in general.

The UV/IR mixing poses a severe problem in the renormalization procedure within perturbation theory since a new type of IR divergences, in addition to the usual UV divergences [6], appears in non-planar diagrams [5]. The finite lattice formulation [7] based on twisted reduced models [8] and their new interpretation [9] in the context of NC geometry, regularizes such divergences — not only the ordinary UV divergences but also the novel IR divergences. It therefore provides a non-perturbative framework to establish the existence of a consistent field theory on a NC geometry. In ref. [10] a simple field theory, 2d U(1) gauge theory on a NC plane[1], has been studied by Monte Carlo simulation, and the existence of a finite continuum limit has been confirmed. The ultraviolet dynamics is described by the commutative 2d U(∞) gauge theory. On the other hand, the infrared dynamics would be described by the commutative 2d U(1) theory, were it not for the UV/IR mixing effects. The "dynamical Aharonov-Bohm effect" observed in this regime clearly demonstrates the non-equivalence to the commutative 2d U(1) theory.

As an interesting physical consequence of the UV/IR mixing in the case of scalar field theory, ref. [13] predicted the existence of a "striped phase", in which non-zero Fourier modes of the scalar field acquire a vacuum expectation value and break the translational invariance spontaneously. This prediction, which was based on a self-consistent Hartree-Fock approximation, was supported later by another study using an effective action [14], and discussed also in the framework of a renormalization group analysis in $4-\varepsilon$ dimensions [15]. Finally, the existence of this new phase was fully established by Monte Carlo simulations [16 – 20]. In ref. [19], in particular, the explicit results for the phase diagram and the dispersion relation were presented in $d=3$. The dispersion relation in the disordered phase shows a positive IR singularity due to the UV/IR mixing, which shifts the energy minimum to a non-vanishing momentum. As one changes the mass parameter in the action, the mode at the energy minimum condenses, which yields the corresponding stripe pattern. The existence of a sensible continuum limit is suggested by the scaling of various physical quantities. In particular the average width of the stripes stabilizes at a finite value in the continuum limit. In even dimensions with the non-commutativity tensor of maximal rank, the emergence of the striped phase can also be conjectured from the eigenvalue distribution of the matrix, which represents the scalar field in NC geometry [18, 21].

In the case of 4d U(1) NC gauge theories, the IR singularity occurs in the one-loop calculation of the vacuum polarization tensor [22]. This changes the dispersion relation of photons at low momenta [23]. In fact the IR singularity in the dispersion relation occurs with an overall *negative* sign [24 – 26] unless one introduces sufficiently many fermionic degrees of freedom. It turns out that the manifestly gauge invariant effective action, which is derived from field theoretical calculations as well as from string theoretical calculations [27, 28], involves the open Wilson line operators [29], which implies that the IR singularity is associated with their condensation. This causes the spontaneous breakdown of translational invariance since the open Wilson lines carry specific non-zero momenta dictated by the

[1] See refs. [11, 12] for related analytic works, including several attempts to actually solve this model.

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star-gauge invariance [29]. In the case of a finite NC torus, the same phenomenon has been understood [30] from its Morita dual *commutative* SU(N) gauge theory with twisted boundary conditions[2].

We should note, however, that these perturbative analyses merely imply that the *perturbative vacuum* of 4d NC gauge theory has tachyonic instability, but they do not exclude the possibility that the system eventually finds a stable vacuum. (An analogous scenario known as the "tachyon condensation" has been proposed in the open string field theory [31][3].) In order to investigate this issue, we definitely need to employ a non-perturbative framework.

In the current work, the lattice formulation [7] is particularly important since it allows us to study the model from first principles by Monte Carlo simulation. We study pure U(1) gauge theory in four dimensions, where two directions are non-commutative, and we find a phase in which open Wilson lines acquire non-zero vacuum expectation values. In fact this phase extends towards weak coupling as the system size is increased, and it turns out that we always end up in this phase in the continuum and infinite-volume limits with a fixed non-commutativity parameter. This is consistent with the prediction from the one-loop calculations. We observe, however, that various physical quantities obey clear scaling behaviors in the above limit, which provides evidence for a possible continuum theory. We also study the dispersion relation. In the weak coupling symmetric phase, it involves a negative IR-singular term, which makes the energy vanish at a non-zero momentum as one approaches the critical point. In the broken phase, although the momentum components in the NC directions are no longer conserved, we can still study the relation between the energy and the momentum in the *commutative* direction. This reveals the existence of the Nambu-Goldstone mode associated with the spontaneous symmetry breakdown.

This paper is organized as follows. In section 2 we define a gauge theory in 4d NC space, and we briefly review the results obtained by perturbative calculations. In section 3 we introduce the lattice formulation and discuss how to take the continuum limit. In section 4 we present the phase diagram of the lattice model. In section 5 we investigate the existence of a sensible continuum limit. In section 6 we study how the space in the NC directions looks like in the broken phase. In section 7 we study the dispersion relation in each phase to gain deeper understanding of the observed phase transition and the continuum limit. Section 8 is devoted to a summary and discussions. In the appendix we explain the algorithm used for our Monte Carlo simulations. Some preliminary results of this work have been presented in proceeding contributions [34].

2. Tachyonic instability in perturbation theory

NC geometry is characterized by the following commutation relation among the space-time

[2]At high tempertature (on the scale of the non-commutativity) the ring diagrams dominate. Their evaluation to all orders implies a critical temperature, above which the magnetic photon mass squared turns negative in the NC planes [32].

[3]This might be more than a plain analogy since the tachyonic instability of NC gauge theory has partial contribution from the closed string tachyon [28, 33].

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coordinates

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\Theta_{\mu\nu}\ , \tag{2.1}$$

where $\Theta_{\mu\nu}$ is the non-commutativity tensor. Here we consider the 4d Euclidean space-time $\hat{x}_\mu$ ($\mu = 1, \cdots, 4$) with the non-commutativity introduced only in the $\mu = 1, 2$ directions, i.e.,

$$\begin{aligned} \Theta_{12} &= -\Theta_{21} = \theta\ , \\ \Theta_{\mu\nu} &= 0 \qquad \text{otherwise}\ . \end{aligned} \tag{2.2}$$

Since we have two commutative directions, we may regard one of them as the Euclidean time. This allows us to alleviate the well-known problems concerning causality and unitarity [35, 36]. Moreover, it also enables us to define the dispersion relation as a useful probe to study the system [19].

Pure U(1) gauge theory in this NC space can be formulated in the path integral formalism by the gauge action

$$\begin{aligned} S &= \frac{1}{4}\int d^4x\, F_{\mu\nu}(x) \star F_{\mu\nu}(x)\ , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i\,g\,(A_\mu \star A_\nu - A_\nu \star A_\mu)\ , \end{aligned} \tag{2.3}$$

where the space-time coordinates x_μ are c-numbers as in ordinary field theories, but the non-commutativity is now encoded in the star product

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\Theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right) f(x)g(y)\bigg|_{x=y}\ . \tag{2.4}$$

The action in eq. (2.3) is invariant under a star-gauge transformation

$$A_\mu \mapsto A_\mu + \partial_\mu\Lambda + ig[A_\mu, \Lambda]_\star\ , \tag{2.5}$$

where the star-commutator is defined by $[f, g]_\star \equiv f \star g - g \star f$. Note that non-linear terms appear in the field strength tensor in eq. (2.3) due to the NC geometry, although we are dealing with a gauge group of rank 1. As a result, the theory shares such properties[4] as a negative beta function [22, 25] with non-Abelian (rather than Abelian) gauge theories in the commutative space.

This model has been studied extensively in perturbation theory. In particular the result that is most relevant for us is the one-loop effective action for the gauge field, which involves the quadratic term [22 – 26]

$$\begin{aligned} \Gamma_{1-\text{loop}} &= \frac{g^2}{\pi^2}\int \frac{d^4p}{(2\pi)^4} A_\mu(p)A_\nu(-p)\frac{\tilde{p}_\mu\tilde{p}_\nu}{|\tilde{p}|^4} + \cdots\ , \\ \text{where}\quad \tilde{p}_\mu &= \Theta_{\mu\nu}p_\nu\ . \end{aligned} \tag{2.6}$$

This term emerges from the non-planar diagrams and represents an effect of UV/IR mixing. Since the effective potential, which is *minus* the effective action according to the adopted

[4]It is also conjectured that NC photons form bound states analogous to glueballs in QCD [37].

convention, involves a *negative* quadratic term, we find that the low momentum modes of the gauge field cause a tachyonic instability.

In fact the quadratic term in (2.6) is invariant under the star-gauge transformation (2.5) only at the leading order in A_μ. In order to obtain a fully gauge invariant effective action, one needs to include terms at higher orders in A_μ as in the supersymmetric case [38]. The term in the gauge-invariant effective action which is most singular at small p is given by [27, 28][5]

$$\Gamma_{1-\mathrm{loop}} = \frac{1}{\pi^2}\int \frac{d^4p}{(2\pi)^4}\, W(p)W(-p)\frac{1}{|\tilde{p}|^4} \ . \tag{2.7}$$

The open Wilson line $W(p)$, which appears here, is a manifestly star-gauge invariant operator defined as [29]

$$W(p) = \int d^d x\, \mathrm{e}^{ip\cdot x}\, \mathcal{P}\exp_\star\left(i\, g\int_x^{x+\tilde{p}} A_\mu(\xi)\, d\xi_\mu\right) \ , \tag{2.8}$$

where $\mathcal{P}\exp$ represents the path-ordered exponential, and the path for the line integral over ξ is taken to be a straight line connecting x and $x+\tilde{p}$. Expanding (2.7) in terms of A_μ, one obtains (2.6) at the leading order, up to an irrelevant constant term. The result (2.7) implies that the instability is associated with the condensation of open Wilson lines $W(p)$ with small p. Since the open Wilson line $W(p)$ carries non-zero momentum, its condensation causes the spontaneous breakdown of translational symmetry.

From the perturbative analysis alone, one cannot tell whether the theory possesses a stable non-perturbative vacuum after the condensation of open Wilson lines. To answer this question and to study the nature of the theory at the stable vacuum, if it exists, we definitely need a fully non-perturbative approach.

3. The lattice model and its continuum limit

3.1 Lattice regularization of NC gauge theory

The lattice regularized version of the theory (2.3) can be defined by an analog of Wilson's plaquette action [7]

$$S = -\beta\sum_x\sum_{\mu<\nu} U_\mu(x)\star U_\nu(x+a\hat{\mu})\star U_\mu(x+a\hat{\nu})^*\star U_\nu(x)^* + \mathrm{c.c.}\ , \tag{3.1}$$

where the symbol $\hat{\mu}$ represents a unit vector in the μ-direction and we have introduced the lattice spacing a. The link variables $U_\mu(x)$ $(\mu = 1,\cdots,4)$ are complex fields on the lattice satisfying the star-unitarity condition

$$U_\mu(x)\star U_\mu(x)^* = U_\mu(x)^*\star U_\mu(x) = 1 \ . \tag{3.2}$$

The star product on the lattice can be obtained by rewriting (2.4) in terms of Fourier modes and restricting the momenta to the Brillouin zone. The action (3.1) is invariant

[5] Refs. [39] pointed out that the appearance of open Wilson lines in the effective action is a generic feature of field theories on NC geometry.

under a star-gauge transformation

$$U_\mu(x) \mapsto g(x) \star U_\mu(x) \star g(x+a\hat{\mu})^* \ , \tag{3.3}$$

where also $g(x)$ obeys the star-unitarity condition

$$g(x) \star g(x)^* = g(x)^* \star g(x) = 1 \ . \tag{3.4}$$

As in the commutative space, one obtains the continuum action (2.3) from (3.1) in the $a \to 0$ limit with the identification $\beta = \frac{1}{2g^2}$ and

$$U_\mu(x) = \mathcal{P} \exp_\star \left(i\, g \int_x^{x+a\hat{\mu}} A_\mu(\xi)\, d\xi_\mu \right) \ . \tag{3.5}$$

In order to study the lattice NC theory (3.1) by Monte Carlo simulations, it is crucial to reformulate it in terms of matrices [7]. In the present setup (2.2), with two NC directions and two commutative ones, the transcription applies only to the NC directions, and the commutative directions remain untouched. Let us decompose the four-dimensional coordinate as $x \equiv (y,z)$, where

$$y \equiv (x_1, x_2) \quad \text{and} \quad z \equiv (x_3, x_4) \tag{3.6}$$

represent two-dimensional coordinates in the NC and in the commutative plane, respectively. We use a one-to-one map between a field $\varphi(x)$ on the four-dimensional $N \times N \times L \times L$ lattice and a $N \times N$ matrix field $\hat{\varphi}(z)$ on a two-dimensional $L \times L$ lattice. This map yields the following correspondence

$$\varphi_1(y,z) \star \varphi_2(y,z) \quad \Longleftrightarrow \quad \hat{\varphi}_1(z)\, \hat{\varphi}_2(z) \ , \tag{3.7}$$

$$\varphi(y + a\hat{\mu}, z) \quad \Longleftrightarrow \quad \Gamma_\mu\, \hat{\varphi}(z)\, \Gamma_\mu^\dagger \ , \tag{3.8}$$

$$\frac{1}{N^2} \sum_y \varphi(y,z) \quad \Longleftrightarrow \quad \frac{1}{N} \operatorname{tr} \hat{\varphi}(z) \ . \tag{3.9}$$

The SU(N) matrices Γ_μ ($\mu = 1,2$), which represent a shift in a NC direction, satisfy the 't Hooft-Weyl algebra

$$\Gamma_1 \Gamma_2 = \mathcal{Z}_{12}\, \Gamma_2 \Gamma_1 \ , \tag{3.10}$$

$$\mathcal{Z}_{12} = \mathcal{Z}_{21}^* = \exp\left(\pi i\, \frac{N+1}{N} \right) \ , \tag{3.11}$$

with the matrix size N being an odd integer. For this particular construction [40, 10, 19], which we are going to use throughout this paper, it turns out that the non-commutativity parameter θ in (2.2) is given by

$$\theta = \frac{1}{\pi} N a^2 \ . \tag{3.12}$$

Note that the extent in the NC directions Na goes to infinity in the continuum limit $a \to 0$ at fixed θ.

Using this map, the link variables $U_\mu(x)$ are mapped to a $N \times N$ matrix field $\hat{U}_\mu(z)$ on the two-dimensional $L \times L$ lattice, and the star-unitarity condition (3.2) simply requires $\hat{U}_\mu(z)$ to be unitary. The action (3.1) can be rewritten in terms of the unitary matrix field $\hat{U}_\mu(z)$ in a straightforward manner. By performing a field redefinition

$$V_\mu(z) \equiv \begin{cases} \hat{U}_\mu(z)\Gamma_\mu & \text{for } \mu = 1,2 \ , \\ \hat{U}_\mu(z) & \text{for } \mu = 3,4 \ , \end{cases} \tag{3.13}$$

where $V_\mu(z) \in \mathrm{U}(N)$, we arrive at

$$\begin{aligned} S &= S_{\mathrm{NC}} + S_{\mathrm{com}} + S_{\mathrm{mixed}} \ , \\ S_{\mathrm{NC}} &= -N\beta\mathcal{Z}_{12} \sum_z \operatorname{tr}\Big(V_1(z)\, V_2(z)\, V_1(z)^\dagger\, V_2(z)^\dagger \Big) + \text{c.c.} \ , \\ S_{\mathrm{com}} &= -N\beta \sum_z \operatorname{tr}\Big(V_3(z)\, V_4(z+a\hat{3})\, V_3(z+a\hat{4})^\dagger\, V_4(z)^\dagger \Big) + \text{c.c.} \ , \\ S_{\mathrm{mixed}} &= -N\beta \sum_z \sum_{\mu=1}^{2} \sum_{\nu=3}^{4} \operatorname{tr}\Big(V_\mu(z)\, V_\nu(z)\, V_\mu(z+a\hat{\nu})^\dagger\, V_\nu(z)^\dagger \Big) + \text{c.c.} \ . \end{aligned} \tag{3.14}$$

These terms represent a plaquette in the NC plane, in the commutative plane and a sum over the plaquettes in the four mixed planes, respectively. Since the action (3.14) is invariant under the gauge transformation

$$V_\mu(z) \mapsto \begin{cases} \hat{g}(z)\, V_\mu(z)\, \hat{g}(z)^\dagger & \text{for } \mu = 1,2 \ , \\ \hat{g}(z)\, V_\mu(z)\, \hat{g}(z+a\hat{\mu})^\dagger & \text{for } \mu = 3,4 \ , \end{cases} \tag{3.15}$$

where $\hat{g}(z) \in \mathrm{U}(N)$, it defines a 2d lattice gauge theory, in which $V_\mu(z)$ ($\mu = 3,4$) correspond to link variables, and $V_\mu(z)$ ($\mu = 1,2$) correspond to scalar fields in the adjoint representation with the specific self-coupling given by S_{NC}. Taking into account the field redefinition (3.13), one easily finds that the $\mathrm{U}(N)$ gauge symmetry (3.15) of the 2d theory corresponds precisely to the star-gauge invariance (3.3) of the 4d NC theory that we started with.

3.2 Eguchi-Kawai equivalence in the planar limit

In fact the lattice model (3.14) can be obtained by the twisted dimensional reduction [8] from 4d pure $\mathrm{U}(N)$ lattice gauge theory[6]. Historically such a model appeared in the context of the Eguchi-Kawai reduction [42], which in the present case provides an explicit relation between the 2d $\mathrm{U}(N)$ theory (3.14) and the 4d $\mathrm{U}(N)$ theory (both in the commutative space) in the large-N limit at fixed β. (This limit is usually referred to as the planar limit since only planar diagrams survive [43].) More specifically, the vacuum expectation value of a Wilson loop defined in each theory coincides in the above limit, if the $\mathrm{U}(1)^2$ symmetry

$$V_\mu(z) \mapsto \mathrm{e}^{i\alpha_\mu} V_\mu(z) \qquad \text{for } \mu = 1,2 \tag{3.16}$$

[6] In the present case, this amounts to considering pure $\mathrm{U}(N)$ gauge theory on a $1 \times 1 \times L \times L$ lattice with the twisted boundary condition [41] for the 1,2 directions, and the periodic boundary condition for the 3,4 directions. The phase factor $\mathcal{Z}_{12}$ in the first term of (3.14) represents the twist.

of the 2d theory (3.14) is not spontaneously broken. We will see that this condition holds in the weak coupling and the strong coupling regions, but it is violated in the intermediate region. In fact this symmetry (3.16) includes the (discrete) translational symmetry of the 4d NC lattice gauge theory in the NC directions as a subgroup up to a star-gauge transformation [44]. Therefore, the spontaneous breaking of the $U(1)^2$ symmetry corresponds precisely to the IR instability of the perturbative vacuum discussed in section 2.

Let us comment on some known results for a totally reduced (one-site) model. The condition for the Eguchi-Kawai equivalence in this case is that the $U(1)^4$ symmetry is not spontaneously broken. The twist was introduced in ref. [8] into the original (untwisted) Eguchi-Kawai model [42] in order to avoid the spontaneous $U(1)^4$ symmetry breaking in the weak coupling region[7]. Indeed the condition is satisfied in the strong coupling and weak coupling expansions, and early Monte Carlo studies suggested that it holds, too, at intermediate couplings [8]. However, Ishikawa and Okawa [47] have recently performed Monte Carlo simulations with much larger N, and observed a signal of spontaneous symmetry breaking at intermediate couplings. This behavior persisted also for choices of the twists other than the minimal one adopted in ref. [8].

The Eguchi-Kawai equivalence, if the condition is met, holds in the large-N limit with a finite lattice spacing a. The continuum limit $a \to 0$ as a commutative 4d $U(\infty)$ gauge theory can be taken in the next step by sending $\beta \to \infty$. In our model, however, the critical β, below which the symmetry breaking occurs, is observed to increase as N^2, which implies that the Eguchi-Kawai equivalence does not persist in the continuum limit. It remains to be seen whether this is the case even if one takes other options in defining twisted reduced models (the choice of the twist, partial or total reduction, etc.). An alternative to ensure the validity of the Eguchi-Kawai equivalence in the continuum limit is the quenched reduced model [48]. Its continuum version [49] has been applied recently [50] to a non-lattice regularization of Matrix Theory [51]. See also refs. [45, 52] for recent developments in the large-N reduction in the continuum.

3.3 Double scaling limit

As one can see from (3.12), the planar limit corresponds to $\theta = \infty$ in the context of NC field theory. In order to take the continuum limit as a NC gauge theory with finite θ, one has to take the $N \to \infty$ and $\beta \to \infty$ limits simultaneously, while satisfying (3.12), which therefore implies $a \to 0$. Following the usual terminology in matrix models, we call it the double scaling limit. Unlike in the planar limit, non-planar diagrams survive and they cause the intriguing UV/IR mixing effects. The inverse coupling constant β has to be tuned in such a way that physical quantities converge. Whether this is possible or not is precisely the issue of (non-perturbative) renormalizability, which we address in the following.

In the 2d NC gauge theory [10], since the $U(1)^2$ symmetry is not spontaneously broken throughout the whole coupling region, the Eguchi-Kawai equivalence holds in the planar

[7]It has been shown recently [46] that in the original (untwisted) Eguchi-Kawai model and in a more general model, the $U(1)^4$ symmetry breaking does not occur at once, but rather in a sequence $U(1)^4 \to U(1)^3 \to U(1)^2 \to U(1) \to$none. Such a partial breaking of $U(1)^4$ symmetry was observed earlier in large N gauge theories in a finite box [45].

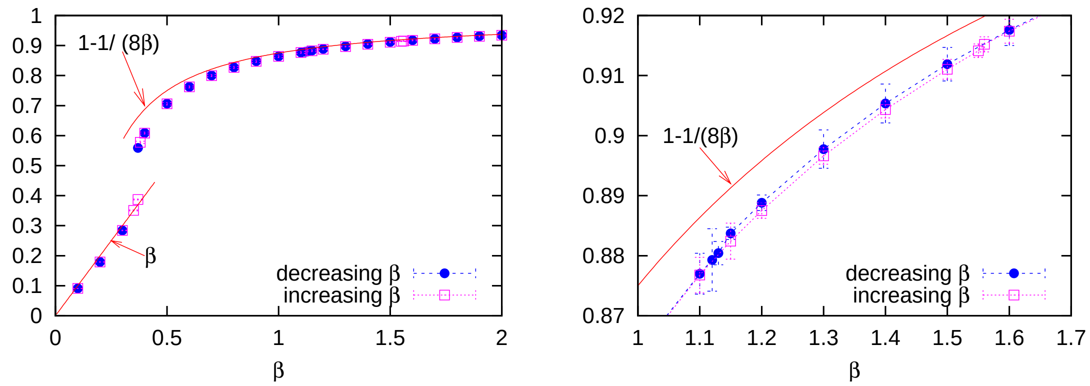

Figure 1: (Left) The expectation value of the normalized action (4.1) is plotted against β for $N = 25$. The two kinds of symbols are used to distinguish the results obtained for decreasing β and increasing β in the "thermal cycle". The solid lines represent the results of the strong and the weak coupling expansions. The discontinuity is observed at $\beta \sim 0.35$. (Right) The zoom-up of the same plot around $1.1 \lesssim \beta \lesssim 1.6$.

limit for all β. The tuning of β after taking the planar limit can therefore be deduced from the exact solution [53] of the 2d $\mathrm{U}(\infty)$ lattice gauge theory. This tuning of β with respect to the lattice spacing a should be used also in the double scaling limit in order to make the correlation functions scale in the UV regime [54]. Whether the scaling extends to the IR regime or not is a non-trivial issue, which was answered affirmatively in ref. [10]. In the present 4d case, since the $\mathrm{U}(1)^2$ symmetry is spontaneously broken in the coupling region relevant to the continuum limit, the tuning of β cannot be deduced from the results in 4d $\mathrm{U}(\infty)$ lattice gauge theory. We therefore have to fine-tune β in such a way that the double scaling is optimized.

4. Phase structure

To begin with, we investigate the phase structure of the lattice model (3.14). Throughout this paper, we take L, the number of sites in the commutative directions, to be $L = N \pm 1$ so that it becomes a multiple of four in order to use four processors on a parallel machine efficiently. The small anisotropy of the lattice should be negligible at large N.

As a standard quantity we plot the normalized action (or the average plaquette)

$$\mathcal{E} = -\frac{1}{12N^2\beta}\langle S\rangle \tag{4.1}$$

against β for $N = 25$ in figure 1. We have performed a "thermal cycle" by carrying out simulations at varying β with the initial configuration taken from a configuration thermalized with a slightly larger/smaller β. There is a gap at $\beta \sim 0.35$ (left), and we observe a slight tendency of a possible hysteresis behavior at $1.1 \lesssim \beta \lesssim 1.6$ (right) although the difference between the two branches, even it exists, is too tiny to be confirmed definitely.

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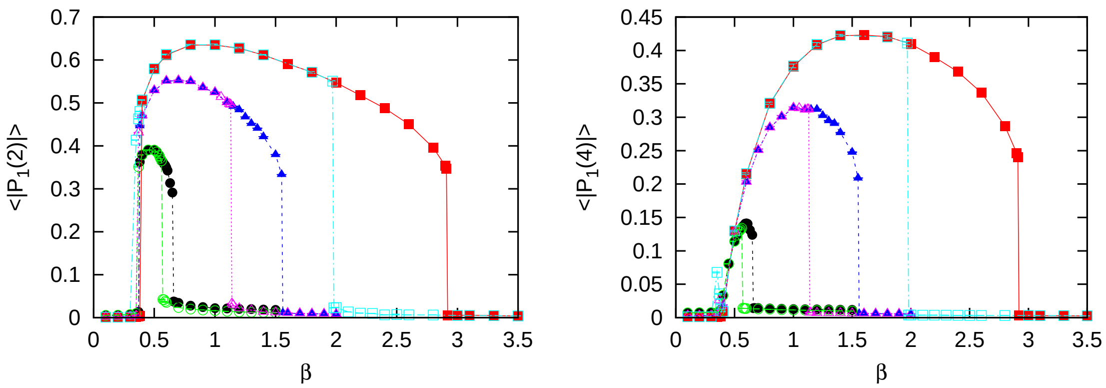

Figure 2: The order parameter $\langle|P_1(n)|\rangle$ is plotted against β for $n = 2$ (left) and $n = 4$ (right). The system size is $N = 15$ (circles), $N = 25$ (triangles) and $N = 35$ (squares). The closed (open) symbols represent results obtained with increasing (decreasing) β, which show a clear hysteresis behavior.

As an order parameter for the spontaneous breaking of the $\mathrm{U}(1)^2$ symmetry (3.16), we define a gauge invariant operator

$$P_\mu(n) = \frac{1}{NL^2} \sum_z \mathrm{tr}\left(V_\mu(z)^n\right) \quad \text{for } \mu = 1, 2 \ , \tag{4.2}$$

which transforms non-trivially under (3.16). This operator corresponds to the open Wilson line (2.8) carrying a momentum with the absolute value [7, 10]

$$p = \frac{2\pi k}{Na} \ , \qquad k = \begin{cases} \frac{n}{2} & \text{for even } n \ , \\ \frac{n+N}{2} & \text{for odd } n \ . \end{cases} \tag{4.3}$$

Since the operator $P_\mu(n)$ with odd n carries a momentum of the cutoff order, it does not couple to excitations that survive in the continuum limit[8]. Therefore we will focus mainly on the even n case in what follows.

In figure 2 we plot $\langle|P_1(n)|\rangle$ against β for $n = 2$ (left) and $n = 4$ (right). Since it turns out that $|P_1(n)|$ and $|P_2(n)|$ are not too different for each configuration[9], we actually take an average over the two NC directions to increase the statistics. We observe that there is a phase, in which the order parameter becomes non-zero. On the other hand, the quantity $\langle|P_1(n)|\rangle$ for *odd* n takes tiny values throughout the whole region of β. This implies that the $\mathrm{U}(1)^2$ symmetry is broken down to $(\mathrm{Z}_2)^2$ in this phase, which we refer to as the "broken phase". The fact that the $\mathrm{U}(1)^2$ symmetry is unbroken, both in the small β regime and

[8]This Z_2 grading of the open Wilson line operators should be regarded as an artifact of the lattice regularization [7].

[9]In principle, it could also be that the $\mathrm{U}(1)^2$ symmetry is broken down only to $\mathrm{U}(1)$. In that case $|P_1(n)|$ and $|P_2(n)|$ would be very different for each configuration, and taking the average over the two directions would not be legitimate. We observed that this is not the case.

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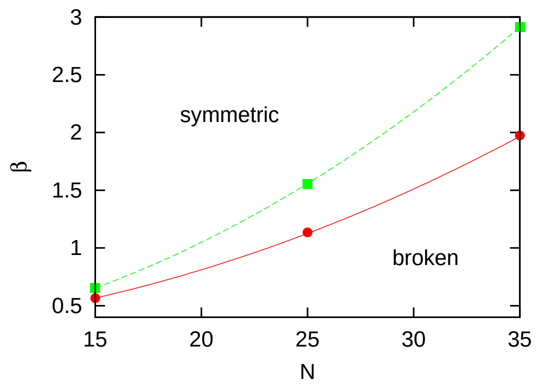

Figure 3: The upper (squares) and lower (circles) critical points between the symmetric phase and the broken phase, obtained from figure 2. The lines represent a fit to $\beta = c_1 N^2 + c_2$, where $c_1 = 0.00226(1)$, $c_2 = 0.145(4)$ for the upper critical point, and $c_1 = 0.00141(1)$, $c_2 = 0.250(5)$ for the lower critical point.

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the large β regime, can be understood from the strong coupling and the weak coupling expansions [8] as in the totally reduced model[10].

The transition between the strong coupling phase and the broken phase occurs at $\beta \sim 0.35$ (for all N), which coincides with the position of the gap in figure 1 (left). Actually this value agrees with the critical point of the bulk transition known in the 4d SU(N) lattice gauge theory with Wilson's plaquette action at large N [55], which is also reproduced by large-N reduced models [8, 56]. Since the $\mathrm{U}(1)^2$ symmetry is unbroken in our model in the strong coupling phase, something must occur at the critical β of the bulk transition from the viewpoint of the Eguchi-Kawai equivalence. What actually happens is that the $\mathrm{U}(1)^2$ symmetry is spontaneously broken down to $(\mathrm{Z}_2)^2$, and the equivalence ceases to hold precisely at that point.

The transition between the broken phase and the weak coupling phase is more important since it is relevant for the continuum limit. The hysteresis behavior, which indicates a first order phase transition, can now be seen clearly, unlike in figure 1. In fact the lower critical point of this phase transition, denoted as β_c in the following, can be estimated by perturbation theory, and we obtain [57] $\beta_c \sim N^2$ at large N, which is consistent with our results shown in figure 3. This implies, in particular, that one always ends up in the broken phase if one takes the $N \to \infty$ limit at fixed β (planar limit), which corresponds to $\theta = \infty$ in the context of NC gauge theory.

Let us next consider closed Wilson loops, which play an important role in commutative gauge theories as a criterion for confinement. In the present case, since we introduce non-commutativity only in two directions, there are three kinds of square-shaped Wilson loops depending on their orientations. Using the parallel transporter

$$\mathcal{V}_\nu(z,n) \equiv V_\nu(z) V_\nu(z + a\hat{\nu}) \cdots V_\nu(z + (n-1)a\hat{\nu}) \tag{4.4}$$

[10] A similar phase structure is obtained in the totally reduced model with the minimal twist [47]. In that case, however, $\langle |P_1(n)| \rangle$ for odd n also acquires a non-zero value. This is not so surprising since the argument based on the momentum spectrum (4.3) does not apply to the model with the minimal twist.

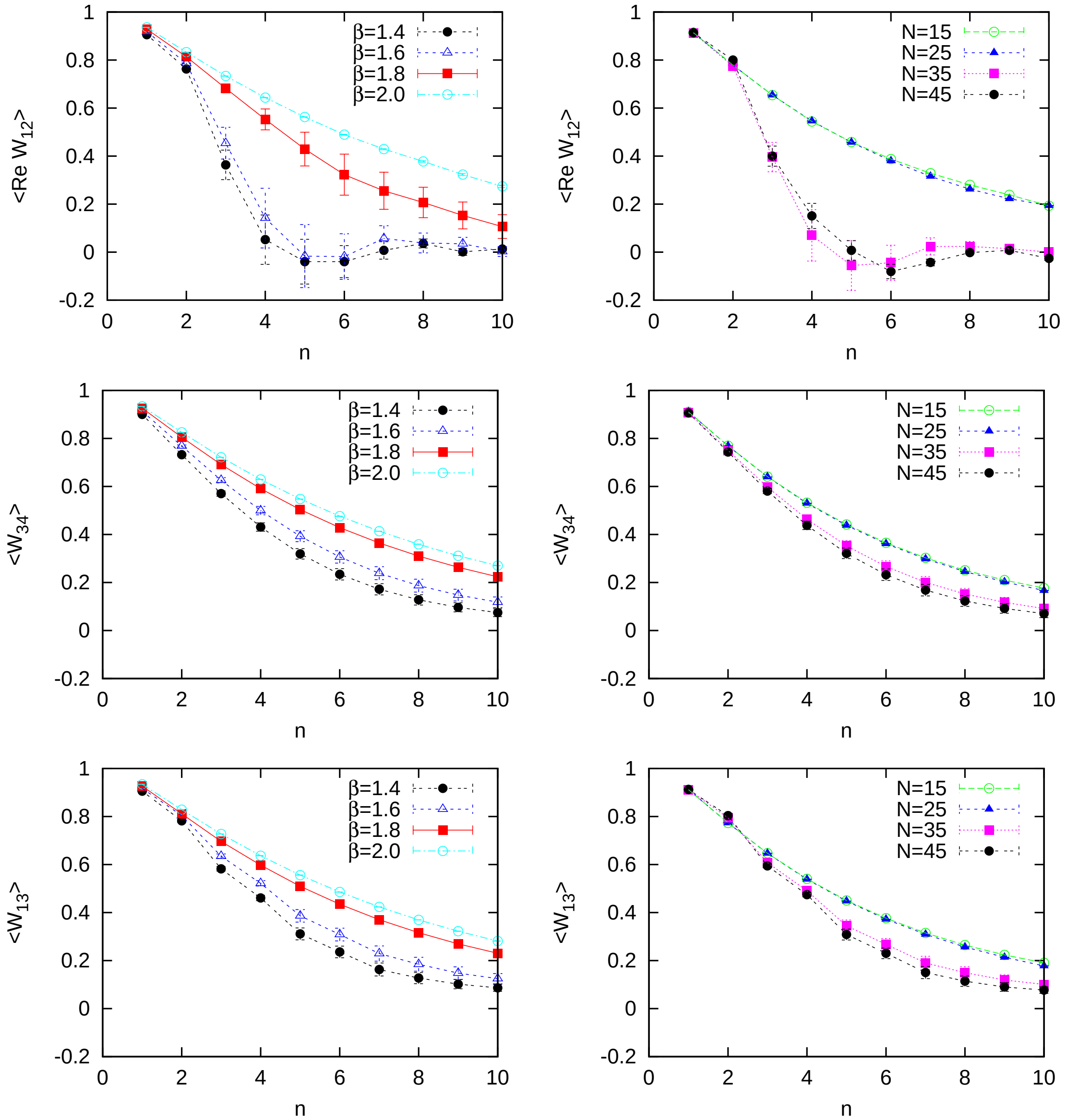

Figure 4: (Left column) The expectation value of the Wilson loop is plotted against its size n for various β at $N = 35$. The results for $\beta = 2.0$ are obtained in the symmetric phase. (Right column) The expectation value of the Wilson loop is plotted against its size n for $N = 15, 25, 35, 45$ at $\beta = 1.5$. The results for $N = 15, 25$ are obtained in the symmetric phase.

in the commutative directions, the closed, square-shaped Wilson loops can be defined as

$$\begin{aligned} W_{12}(n) &= (\mathcal{Z}_{12})^{n^2} \frac{1}{NL^2} \sum_z \mathrm{tr}\left(V_1(z)^n \, V_2(z)^n \, V_1(z)^{\dagger n} \, V_2(z)^{\dagger n} \right) , \\ W_{\mu\nu}(n) &= \frac{1}{NL^2} \sum_z \mathrm{tr}\left(V_\mu(z)^n \, \mathcal{V}_\nu(z,n) \, V_\mu(z + na\hat{\nu})^{\dagger n} \, \mathcal{V}_\nu(z,n)^\dagger \right) , \\ W_{34}(n) &= \frac{1}{NL^2} \sum_z \mathrm{tr}\left(\mathcal{V}_3(z,n) \, \mathcal{V}_4(z + na\hat{3}, n) \mathcal{V}_3(z + na\hat{4}, n)^\dagger \, \mathcal{V}_4(z,n)^\dagger \right) , \end{aligned} \tag{4.5}$$

where $\mu = 1, 2$ and $\nu = 3, 4$. We may define $W_{\mu\nu}(n)$ for $\mu > \nu$ in a similar manner, but it suffices to consider (4.5) since $W_{\mu\nu}(n) = W_{\nu\mu}(n)^*$. Mapping the expression (4.5) back to the 4d NC space, we find that the operator $W_{\mu\nu}(n)$ represents a closed $n \times n$ Wilson loop, whose starting point is integrated over the 4d space. Therefore it does not carry non-zero momentum unlike the open Wilson line $P_\mu(n)$. This corresponds to the fact that $W_{\mu\nu}(n)$ is invariant under the $\mathrm{U}(1)^2$ transformation (3.16), whereas $P_\mu(n)$ is not.

Let us consider the expectation value $\langle W_{\mu\nu}(n)\rangle$. First we reduce the number of observables using discrete symmetries. Due to the symmetry under exchanging the third and the fourth directions, we have $\langle W_{34}\rangle = \langle W_{43}\rangle \in \mathbb{R}$ and $\langle W_{\mu 3}\rangle = \langle W_{\mu 4}\rangle$, where $\mu = 1, 2$. Based on the symmetry under $V_1(z) \mapsto V_2(z)$, $V_2(z) \mapsto V_1(z)^\dagger$, we obtain $\langle W_{1\nu}\rangle = \langle W_{2\nu}\rangle \in \mathbb{R}$, where $\nu = 3, 4$. It then follows that $\langle W_{\mu\nu}\rangle$ with $\mu = 1, 2$ and $\nu = 3, 4$ are all real and equal. We will use $\langle W_{13}\rangle$ as a representative, but in actual simulation we measure $\frac{1}{4}\sum_{\mu=1}^{2}\sum_{\nu=3}^{4}\langle \mathrm{Re}\, W_{\mu\nu}\rangle$ to increase the statistics. Note that $\langle W_{12}\rangle$, which represents a closed Wilson loop in the NC directions, is complex in general unlike in commutative gauge theories. This occurs because the non-commutativity tensor $\Theta_{\mu\nu}$ breaks parity.

In figure 4 (left column) the expectation value of the Wilson loop is plotted against its linear size n for various β values at $N = 35$. First let us focus on the results at $\beta = 2.0$, which are obtained in the symmetric phase. Here the three types of Wilson loop are almost identical and follow the perimeter law over the whole range of n shown in the figure. This implies that the system is qualitatively similar to 4d compact U(1) lattice gauge theory in the commutative space, which is non-confining at weak coupling. Note, however, that the effect of the NC geometry is seen in the dispersion relation for the same β (figure 11). As we decrease β, the system enters the broken phase, and the three kinds of Wilson loops start to drift apart.

The right column of figure 4 shows the results for various N at $\beta = 1.5$. The data for $N = 15$ and $N = 25$, which are obtained in the symmetric phase, lie almost on top of each other. This can be interpreted as the scaling behavior corresponding to the would-be planar limit in the symmetric phase, although one actually enters the broken phase as N is increased further. In fact the effect of increasing N is similar to that of decreasing β, which suggests the possibility to make the results scale by increasing β and N simultaneously.

5. Existence of a continuum limit

In this section we investigate whether it is possible to tune β as a function of N in such a way that physical quantities scale. For the rest of this paper except for figure 9, we set $a = 1$ for $N = 45$ as a convention, and the lattice spacing a for other N is determined[11] through (3.12) with $\theta = \frac{45}{\pi} \simeq 14.3$.

As a practical strategy to fine-tune β, we focus on the expectation value of the square Wilson loop $W_{34}(n)$ in the commutative directions since it has a smooth dependence on

[11] If we had discarded the constraint (3.12) and determined a for each N to optimize the scaling, we would have concluded that θ *had to be fixed* in order to obtain the scaling behavior. This follows from the fact that the scaling functions obtained for fixed θ have non-trivial θ-dependence. That θ does not receive renormalization should therefore be considered as an observation rather than an input of our study.

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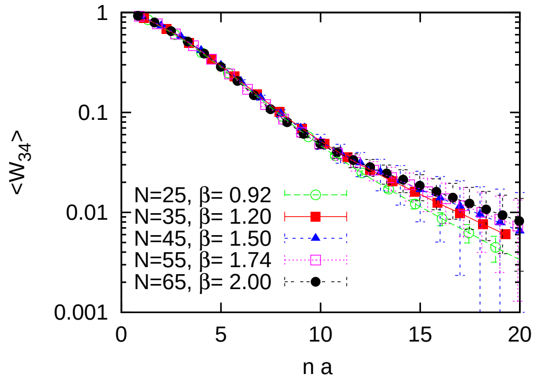

Figure 5: The expectation value of the Wilson loop in the commutative plane. By tuning β depending on N, we can make the results scale.

N	a	β
25	1.34	0.92
35	1.13	1.20
45	1.00	1.50
55	0.90	1.74
65	0.83	2.00

Table 1: The sets of parameters used for the double scaling limit.

its size n. Table 1 shows the optimal values of β for each N, and figure 5 shows the corresponding plot. The horizontal axis represents the physical size (na) of the loop. We observe a clear scaling behavior, and the scaling region extends as N increases. In fact the optimal β increases with N much slower than the lower critical point β_c between the broken phase and the weak coupling phase, which grows as N^2 (see figure 3). This implies that we remain in the broken phase in the double scaling limit.

Let us see whether other quantities scale as well using the *same* sets of parameters as those given in table 1. In figure 6 we plot the expectation value of the Wilson loop in the NC plane (left) and in the mixed planes (right). We do observe a compelling scaling behavior.

The Wilson loop in the commutative plane follows the perimeter law at large size, which suggests that the theory is non-confining in the commutative directions. The Wilson loop in the NC plane is complex in general, and its real part oscillates around zero. In figure 7 we plot the absolute value and the phase of the Wilson loop against the physical area $A = (na)^2$. The absolute value decreases monotonously obeying roughly the area law. This is not so surprising since non-Abelian nature comes in through the star product in (2.3), although we are dealing with a U(1) theory. The phase Φ grows linearly as $\Phi = \frac{1}{\theta}A$, which is reminiscent of the Aharonov-Bohm effect with the magnetic field $B = \theta^{-1}$ [4]. The same behavior has been observed in the 2d case [10].

The behavior of the Wilson loop in the mixed directions is somehow between the

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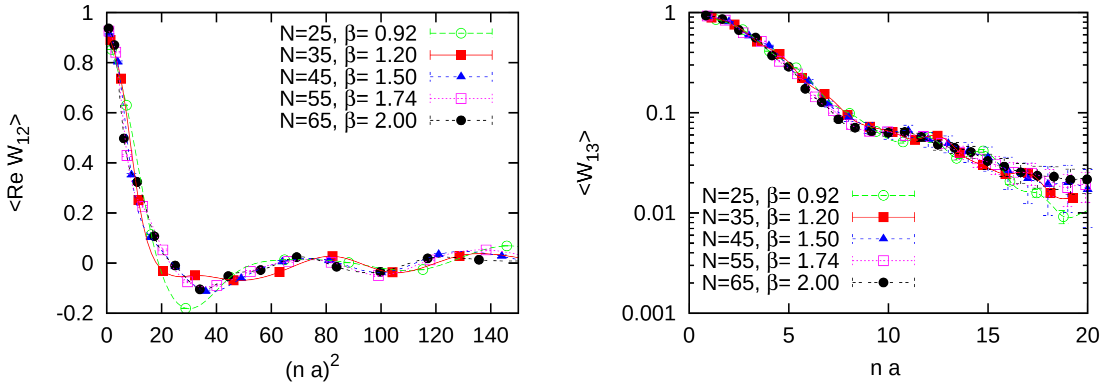

Figure 6: The expectation value of the Wilson loop in the NC plane (left) and in the mixed planes (right). The values of β are given in table 1, which was chosen in such a way that the expectation value of the Wilson loop in the *commutative* plane scales.

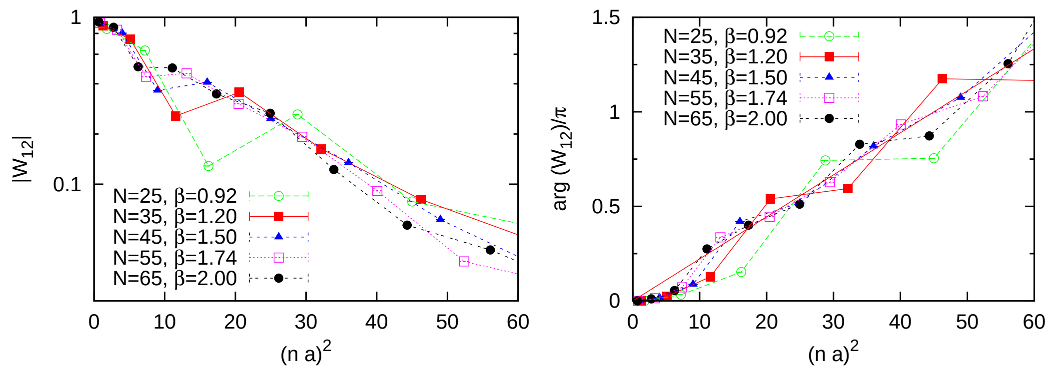

Figure 7: The absolute value (left) and the phase (right) of the Wilson loop in the NC plane is plotted against the physical area $A = (na)^2$. The absolute value follows roughly an area law. Beyond small areas, the phase Φ agrees with the Aharonov-Bohm like behavior $\Phi = A/\theta$ represented by the solid straight line.

other two kinds. It decreases following roughly the perimeter law for large size, but a slight oscillating behavior seems to be superimposed (although these Wilson loops are real). This suggests that the shape of the potential between a quark and an anti-quark separated in a NC direction is oscillating, but we do not have a clear physical interpretation of such a behavior.

Let us turn to the open Wilson line, which was studied in section 4 to investigate the spontaneous breakdown of $\mathrm{U}(1)^2$ symmetry. In figure 8 (left) we plot the expectation value $\langle |P_1(n)| \rangle$ for even n against the momentum defined by (4.3). We observe a tendency that the results lie on a single curve except for the $n = 2$ data (corresponding to the point at the smallest p for each N). A similar anomalous behavior is seen also in figure 7 for $n = 3$ data. We consider these behaviors to be finite N artifacts since they tend to disappear with increasing N.

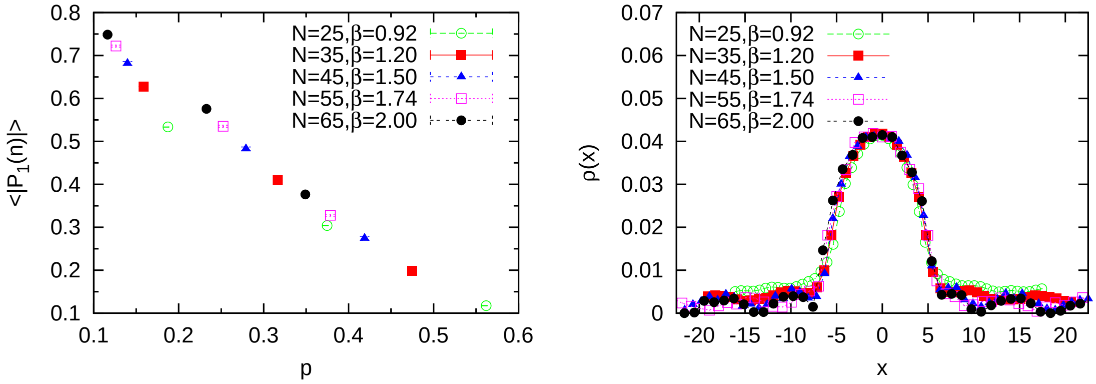

Figure 8: (Left) The expectation value $\langle|P_\mu(n)|\rangle$ for even n is plotted against the momentum $p = \frac{\pi}{Na}n$ for the parameters listed in table 1. (Right) The eigenvalue distribution $\rho(x)$ is plotted for the parameters listed in table 1.

6. Extent of the dynamical NC space

Since the translational invariance in the NC directions is spontaneously broken in the broken phase, it is natural to ask how the space actually looks like in those directions.

As a related quantity, let us consider the eigenvalues of the unitary matrix $V_\mu(z)$ $(\mu = 1, 2)$, which we denote as $e^{i\vartheta_{\mu j}(z)}$ $(j = 1, \cdots, N)$, where $-\pi < \vartheta_{\mu j}(z) \leq \pi$. The open Wilson line (4.2) can be written in terms of the eigenvalues as

$$P_\mu(n) = \frac{1}{NL^2} \sum_z \sum_{j=1}^{N} \mathrm{e}^{in\vartheta_{\mu j}(z)} \ . \tag{6.1}$$

Since the $\mathrm{U}(1)^2$ transformation (3.16) rotates all the eigenvalues by a constant angle α_μ, the eigenvalue distribution should be uniform if the $\mathrm{U}(1)^2$ symmetry is not spontaneously broken. This is how the open Wilson line serves as an order parameter. In the broken phase, we have seen that the open Wilson line $P_\mu(n)$ acquires a non-zero expectation value only for even n. This implies that the eigenvalues are clustered in two bunches in a Z_2-symmetric way. Since the translation in the NC direction is represented by the $\mathrm{U}(1)^2$ transformation, we may consider that the eigenvalue distribution represents the "shape" of the NC space.

Let us define the eigenvalue distribution by

$$\rho(x) \equiv \frac{1}{2NL^2} \sum_z \sum_{\mu=1}^{2} \sum_{j=1}^{N} \left\langle \delta\left(x - \frac{Na}{\pi}\vartheta_{\mu j}(z)\right)\right\rangle \ , \tag{6.2}$$

where we have taken an average over the two NC directions as we did for $\langle|P_1(n)|\rangle$. The coefficient of $\vartheta_{\mu j}(z)$ is motivated from the corresponding relation $p = \frac{\pi}{Na}n$ (n : even) for the momentum conjugate to the coordinate x. Before taking an ensemble average, we rotate each configuration according to (3.16) in such a way that $\sum_z \mathrm{tr}\, V_\mu(z)^2$ becomes real positive[12]. Since the eigenvalue distribution $\rho(x)$ is invariant under the shift $x \mapsto x + Na$

[12]We average over the Z_2-ambiguity in fixing the angle.

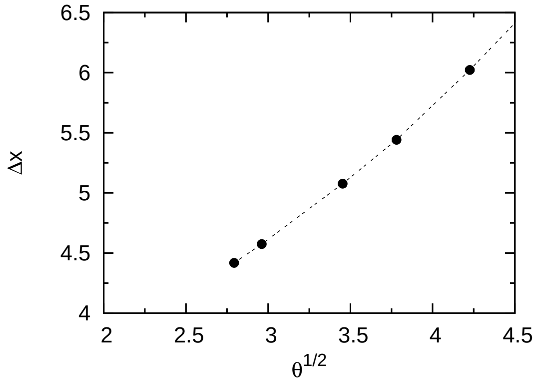

Figure 9: The extent of the dynamical space in the NC directions defined by (6.4) from the view point of string theory, is plotted against $\sqrt{\theta}$ for $N = 35$.

modulo $2Na$, we may restrict ourselves to the fundamental domain $|x| \leq Na/2$. Figure 8 (right) demonstrates a clear scaling behavior of $\rho(x)$. The eigenvalue distribution can be interpreted as the density distribution of D-branes [27]. If we take the view point of string theory, in which the space-time is represented by the eigenvalue distribution of the matrices [61, 62], our results imply that the "dynamical space" in the NC directions has shrunk, but it has a finite extent in the double scaling limit.

Since we have seen that the extent of the "dynamical space" in the NC direction is finite for a fixed θ, it is natural to ask how the extent depends on θ. Let us note that

$$\begin{aligned}\frac{1}{2}\sum_{\mu=1}^{2}\langle|P_\mu(2)|\rangle &= 2\int_{-Na/2}^{Na/2} dx\, e^{\frac{2\pi i}{Na}x}\rho(x)\\ &= 1 - \left(\frac{2\pi}{Na}\right)^2 \int_{-Na/2}^{Na/2} dx\, x^2 \rho(x) + \cdots \ , \qquad (6.3)\end{aligned}$$

where taking the absolute value on the left-hand side corresponds to rotating $\vartheta_{\mu j}(z)$ in the definition (6.2) of $\rho(x)$ before taking the ensemble average as explained above. In the last line, we have assumed that $\rho(x)$ is peaked around $x = 0$ in the fundamental domain $|x| \leq Na/2$. As a definition for the extent of the dynamical space in the NC direction, we therefore use

$$\Delta x \equiv \frac{Na}{\pi}\sqrt{\frac{1}{2}\left(1 - \frac{1}{2}\sum_{\mu=1}^{2}\langle|P_\mu(2)|\rangle\right)} \ . \qquad (6.4)$$

It should be mentioned that the space-time, on which the NC gauge theory is defined, has the extent Na, which diverges in the double scaling limit. However, since the translational invariance in the NC directions is spontaneously broken, the observer on the NC space-time does not recognize that the space-time extends to infinity. The quantity Δx measures the extent of the space-time in the NC directions, which is recognized by the observer to be qualitatively uniform. In that sense, Δx is analogous to the width of the stripes [19] in the 3d NC $\lambda\phi^4$ theory, in which the translational invariance in the NC directions is broken due to the space-dependent vacuum expectation value of the scalar field.

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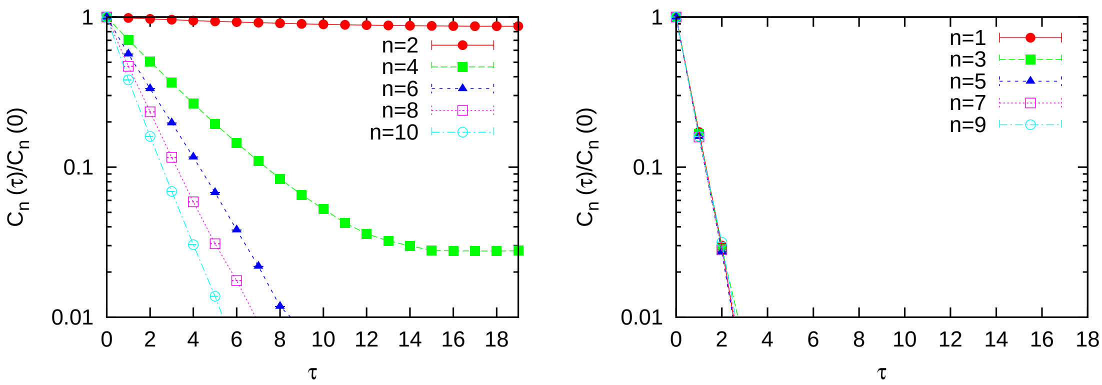

Figure 10: Two-point correlation function of the open Wilson lines for even n (left) and odd n (right) is plotted against the separation τ for $N = 35$ and $\beta = 2.00$. The vertical axis is normalized in such a way that the function starts off from unity at the origin.

Instead of repeating the whole procedure of taking the double scaling limit at each θ, here we assume that the double scaling is obtained for the same sets[13] of (a, β) listed in table 1. This assumption is justified at large N given that the ultraviolet properties of NC theories are independent of θ. For various values of β in the range $0.92 \le \beta \le 2.00$, we determine a by interpolating the relation between a and β presented in table 1. The value of θ is then determined from (3.12) using a and N. In figure 9 we plot Δx against $\sqrt{\theta}$ for $N = 35$. At large θ the extent Δx increases linearly with $\sqrt{\theta}$ as expected on dimensional account.

7. Dispersion relation

In this section we investigate the dispersion relation in the symmetric phase and the broken phase separately. The analysis in the symmetric phase reveals the IR singularity (2.6), which is responsible for the phase transition, whereas the analysis in the broken phase enables us to identify the Nambu-Goldstone mode associated with the spontaneous breakdown of the $\mathrm{U}(1)^2$ symmetry. Thanks to the existence of the commutative directions in the present setup, we may regard one of the coordinates (say, x_4) as "time". From the exponential decay of the two-point correlation function of open Wilson line operators separated in the "time" direction, we can extract the energy of a state that couples to the operator. Similar studies have been done also in the case of NC scalar field theory [19].

7.1 Results in the symmetric phase — IR singularity

Let us define the open Wilson line operator at a fixed time x_4 as

$$P_\mu(x_4, n) \equiv \frac{1}{NL} \sum_{x_3} \mathrm{tr}\left(V_\mu(x_3, x_4)^n\right) \quad \text{for } \mu = 1, 2 \ , \tag{7.1}$$

[13] In fact one can always multiply the lattice spacing a by a θ-dependent factor without affecting the scaling property. This ambiguity corresponds to the arbitrary choice of the Λ-parameter at each θ. The assumption we adopted here corresponds to taking the Λ-parameter independent of θ.

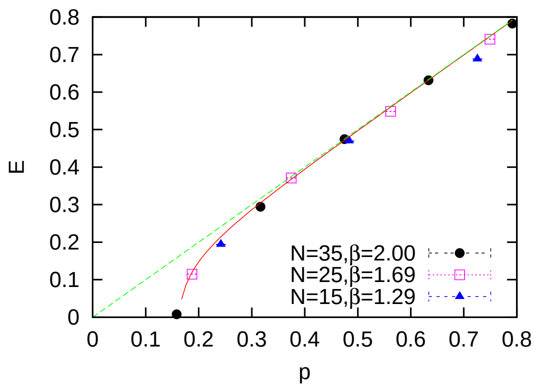

Figure 11: The dispersion relation in the symmetric phase. The energy E obtained from the two-point correlation function (7.2) is plotted against the momentum p for $(N, a, \beta) = (35, 1.13, 2.00)$, $(25, 1.32, 1.69)$, $(15, 1.73, 1.29)$.

which has a zero momentum component in the x_3 direction[14] and a non-zero momentum component (4.3) in a NC direction depending on n. Then we define the two-point correlation function of the open Wilson lines

$$C_n(\tau) \equiv \frac{1}{2} \sum_{\mu=1}^{2} \sum_{x_4} \Big\langle P_\mu(x_4, n)^* \cdot P_\mu(x_4 + \tau, n) \Big\rangle \tag{7.2}$$

with a separation τ in the temporal direction. In actual measurement we also consider the case with the roles of x_3 and x_4 exchanged, and take an average over the two cases to increase the statistics.

In figure 10 we plot the two-point correlation function for even n (left) and odd n (right). For even n we observe clear exponential behaviors $\mathrm{e}^{-\lambda\tau}$, from which we extract the energy $E = \lambda/a$ at each momentum $p = \frac{\pi}{Na} n$. For odd n the decay is very rapid, which suggests that the energy (as well as the momentum) is on the cutoff scale.

In figure 11 we show the dispersion relation obtained from the two-point correlation function (7.2) for even n. The set of parameters (N, a, β) is chosen as in the broken phase. Namely, a is determined through (3.12) with $\theta = \frac{45}{\pi} \simeq 14.3$, and we fine-tune β at each N in such a way that the scaling behavior of the Wilson loops in the commutative plane is optimized. It turns out that the data points (E, p) for different N lie to a good approximation on a single curve

$$E^2 = p^2 - \frac{c}{(\theta p)^2} \tag{7.3}$$

with $c \simeq 0.1285$. This form of the dispersion relation is expected from the one-loop calculation of the vacuum polarization [23–26]. Due to the negative sign of the second term in (7.3), the usual Lorentz invariant (massless) dispersion relation is bent down. The IR singularity is regularized on the finite lattice since the smallest non-zero momentum (which

[14] One can introduce a non-zero momentum component p_3 in the x_3 direction by inserting a phase factor $\mathrm{e}^{ip_3x_3}$ in the summation over x_3 in eq. (7.1). The dispersion relation (7.3) will then have a term $(p_3)^2$ on the right-hand side.

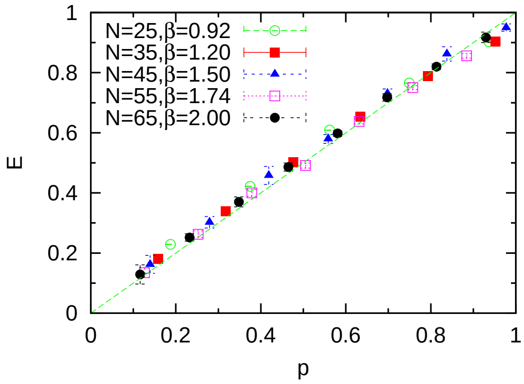

Figure 12: The dispersion relation in the broken phase. The energy E obtained from the two-point correlation function (7.2) is plotted against the momentum p for the parameters listed in table 1.

corresponds to $n = 2$) is given by $\frac{2\pi}{Na} \propto \frac{1}{\sqrt{N}}$. However, if one attempts to increase N further, the energy at the smallest non-zero momentum vanishes at some N, and one enters the broken phase. Therefore we cannot take the double scaling limit in the symmetric phase. The scaling observed in the symmetric phase represents an effective theory with a finite cutoff. In the case of NC scalar field theory [19], the double scaling limit can be taken in the symmetric phase since the IR singularity appears in the dispersion relation with the positive sign.

7.2 Results in the broken phase — Nambu-Goldstone mode

When we extend the study of the dispersion relation to the broken phase, we should note that the momentum components in the NC directions are no longer conserved, although the momentum component in the *commutative* direction still is.

Let us therefore define the open Wilson line operator at a fixed time x_4 carrying momentum p in the commutative direction as

$$\tilde{P}_\mu(x_4, p) \equiv \frac{1}{NL} \sum_{x_3} e^{-ipx_3} \mathrm{tr}\left(V_\mu(x_3, x_4)^2\right) \tag{7.4}$$

for $\mu = 1, 2$. We have chosen the power of V_μ to be the smallest even number so that the operator couples most effectively to the Nambu-Goldstone mode associated with the spontaneous breakdown of the U$(1)^2$ symmetry. Then we define the two-point correlation function of the open Wilson lines

$$\tilde{C}_p(\tau) \equiv \frac{1}{2} \sum_{\mu=1}^{2} \sum_{x_4} \left\langle \tilde{P}_\mu(x_4, p)^* \cdot \tilde{P}_\mu(x_4 + \tau, p) \right\rangle \tag{7.5}$$

with a separation τ in the temporal direction. As in the symmetric phase, we also consider the case with the roles of x_3 and x_4 exchanged, and take an average over the two cases to increase the statistics.

We measure the two-point correlation function and extract the energy at each momentum p. The result is shown in figure 12. It is consistent with the expected massless behavior $E = p$. The discrepancies observed at both ends of the spectrum may be interpreted as finite volume effects and finite lattice-spacing effects, respectively, and they tend to disappear as N increases.

In the 3d NC $\lambda\phi^4$ theory studied in ref. [19], we only have pseudo Nambu-Goldstone modes, since the translational symmetry (which is spontaneously broken) is discretized on the lattice. The corresponding energy therefore vanishes only in the continuum limit. In the present case of gauge theory, the translational symmetry is enhanced to the continuous $\mathrm{U}(1)^2$ symmetry even on the lattice. Therefore, the energy corresponding to the Nambu-Goldstone mode *at zero momentum* is exactly zero even before taking the continuum limit. This motivated us to introduce a non-zero momentum component p in the commutative direction, which makes the energy non-zero. (Note also that for $p = 0$, we need to subtract the disconnected part, which amplifies numerical uncertainties.) The Nambu-Goldstone mode can still be identified by studying the dispersion relation.

8. Summary and discussions

In this paper we studied four-dimensional gauge theory in NC geometry from first principles based on its lattice formulation. In particular we clarified the fate of the tachyonic instability encountered in perturbative calculations. The lattice formulation is suited for such a study since the IR singularity responsible for the instability is regularized in a star-gauge invariant manner, and we can trace the behavior of the system as the regularization is removed. This revealed the existence of a first order phase transition associated with the spontaneous breakdown of the $\mathrm{U}(1)^2$ symmetry, which includes the translational symmetry in the NC directions as a discrete subgroup.

In the weak coupling symmetric phase, we studied the dispersion relation and confirmed the presence of an IR singularity. This IR singularity prevents us from taking the continuum limit in the symmetric phase. While we are able to observe scaling up to some finite N, we cannot increase N further since the energy of the lowest momentum mode vanishes, and the vacuum becomes unstable. In the broken phase, however, we provided evidence for a sensible continuum limit. The phase is characterized by the condensation of open Wilson lines, which represent the "tachyon" in the unstable perturbative vacuum. By studying the dispersion relation in the broken phase, we confirmed the appearance of the Nambu-Goldstone mode associated with the spontaneous breakdown of the $\mathrm{U}(1)^2$ symmetry.

By measuring the eigenvalue distribution of V_μ in the NC directions, we find that the dynamical space in the NC directions has shrunk, but it has a finite *physical* extent in the continuum limit. An analogous first order phase transition is found in gauge theories on fuzzy manifolds [58], where the fuzzy manifolds collapse at sufficiently large couplings. The instability in those cases is due to the uniform condensation of a scalar field on the fuzzy manifold[15]. The phenomenon that the space-time itself becomes a dynamical object is characteristic to gauge theories on NC geometry. This should be closely related to the dynamical compactification of extra dimensions[16] in string theory [61, 62] based on the

[15]The fuzzy manifolds can be stabilized by adding a sufficiently large mass to the scalar field [59].

[16]As a closely related line of research, see refs. [60], which use fuzzy spheres for compactified dimensions.

IKKT matrix model [63]. Let us recall that the NC geometry appears in string theory as a result of introducing a background tensor field [4]. Our conclusion therefore suggests that the dynamical compactification in string theory may be associated with the spontaneous generation of the background tensor field in six dimensions. This is reminiscent of the spontaneous magnetization in 4d non-Abelian gauge theory at finite temperature [64] or in 3d QED [65].

On the other hand, if we wish to obtain a phenomenologically viable *4d model*, we may stay in the symmetric phase by keeping the UV cutoff finite and view the NC gauge theory as an effective theory of a more fundamental theory. A θ-deformed dispersion relation for the photon such as the one displayed in figure 11 should then have implications [66] on observational data from blazars (highly active galactic nuclei) [67], which are assumed to emit bursts of photons simultaneously, covering a broad range of energy. Such experimental efforts will be intensified in the near future. For instance, the Gamma-ray Large Area Space Telescope (GLAST) project [68] is scheduled to be launched in September 2007 and to monitor gamma rays from 20 MeV up to 1 TeV. In particular, a relative delay of these photons depending on the frequency [69] could hint at a NC geometry.

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A. The algorithm for the Monte Carlo simulation

In this section we describe the algorithm used to simulate the model (3.14). The main part of the simulation is performed by the heat-bath algorithm generalizing ref. [70], where $V_\mu(z)$ is updated by multiplying a SU(N) matrix. In order to implement the integration over the U(N) — instead of SU(N) — group manifold appropriately, we also need to include a Metropolis procedure for updating $V_\mu(z)$ by multiplying a phase factor[17]. These procedures, described in the following two subsections, define "one sweep" in our simulation. Measurements have been performed every 50 sweeps. The number of configurations used to obtain an ensemble average is 1980, 1940, 420, 696, 830 for $N = 25, 35, 45, 55, 65$, respectively, for the sets of parameters in table 1.

[17]For instance, if we used the heat-bath algorithm alone, $\det V_\mu(z)$ would not change during the simulation. The Metropolis procedure would be unnecessary if the model (3.14) were an SU(N) gauge theory instead of U(N). The difference of U(N) and SU(N) is irrelevant in the planar limit, but not necessarily in the double scaling limit.

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A.1 Heat-bath algorithm for multiplying a SU(N) matrix

Since two terms, $S_{\rm NC}$ and $S_{\rm mixed}$, make the action (3.14) non-linear with respect to $V_\mu(z)$ for each μ and z, we cannot apply the heat-bath method [71] as it stands. Note, however, that $S_{\rm NC}$ is nothing but the action for the twisted Eguchi-Kawai model for each z, which can be linearized following ref. [70]. Namely we introduce an auxiliary field $Q_{12}(z)$, which is a general complex $N\times N$ matrix, and consider the action

$$\begin{aligned} S'_{\rm NC} = N\beta \sum_z \Big[& {\rm tr} \left\{ Q_{12}(z)^\dagger Q_{12}(z) \right\} \\ & -2\,{\rm Re}\,{\rm tr} \left\{ Q_{12}(z)^\dagger \Big(t\, V_1(z)\, V_2(z) + t^*\, V_2(z)\, V_1(z) \Big) \right\} \Big] \ , \end{aligned} \tag{A.1}$$

where t is a square root of $\mathcal{Z}_{12}$. Completing the squares and integrating out the auxiliary field $Q_{12}(z)$, we retrieve $S_{\rm NC}$.

We use a similar trick to linearize $S_{\rm mixed}$ in (3.14). Namely we introduce the auxiliary fields $Q_{13}(z)$, $Q_{14}(z)$, $Q_{23}(z)$ and $Q_{24}(z)$, each of which is a general complex $N\times N$ matrix, and consider the action

$$\begin{aligned} S'_{\rm mixed} = N\beta \sum_z \sum_{\mu=1}^{2} \sum_{\nu=3}^{4} \Big[& {\rm tr} \left(Q_{\mu\nu}(z)^\dagger Q_{\mu\nu}(z) \right) \\ & -2\,{\rm Re}\,{\rm tr} \left\{ Q_{\mu\nu}(z)^\dagger \Big(V_\mu(z) V_\nu(z) + V_\nu(z) V_\mu(z+a\hat{\nu}) \Big) \right\} \Big] \ . \end{aligned} \tag{A.2}$$

Completing the squares and integrating out the auxiliary fields, we retrieve $S_{\rm mixed}$.

Since the new action

$$S' = S'_{\rm NC} + S_{\rm com} + S'_{\rm mixed} \tag{A.3}$$

is linear with respect to $V_\mu(z)$, we can use the heat-bath algorithm to update $V_\mu(z)$ by multiplying a matrix in one of the $N(N-1)/2$ SU(2) subgroups of the SU(N) [72]. We repeat this procedure for all the independent SU(2) subgroups. The auxiliary fields can be easily updated by generating gaussian variables and shifting them appropriately depending on the $V_\mu(z)$ field.

A.2 Metropolis algorithm for multiplying a phase factor

The updating procedure for rotating the phase of $V_\mu(z)$ is implemented using the Metropolis algorithm. It turns out that we can increase the acceptance rate by rotating the phase of the auxiliary field $Q_{\mu\nu}(z)$ in a covariant manner as

$$\left\{ \begin{array}{ll} V_\mu(z) & \to \mathrm{e}^{i\alpha_\mu(z)} V_\mu(z) \\ Q_{12}(z) & \to \mathrm{e}^{i\alpha_\mu(z)} Q_{12}(z) \ , \end{array} \right. \qquad \left\{ \begin{array}{ll} V_\nu(z) & \to \mathrm{e}^{i\alpha_\nu(z)} V_\nu(z) \\ Q_{\mu\nu}(z) & \to \mathrm{e}^{i\alpha_\nu(z)} Q_{\mu\nu}(z) \ , \end{array} \right. \tag{A.4}$$

where $\mu=1,2$ and $\nu=3,4$. (No sum is taken in (A.4) even if an index appears in a term twice.) At each step, the transformation parameter $\alpha_\mu(z)$ is non-zero for particular μ and z, and it is given by a uniform random number within the range $[0, \frac{2\pi\epsilon}{N}]$, where $\epsilon = 0.1$ was chosen to keep the acceptance rate reasonably high (e.g., 60%). We accept the trial configuration with the probability $\min(1, \mathrm{e}^{-\Delta S'})$, $\Delta S' \equiv S'_1 - S'_0$, where S'_1 and S'_0 are the action evaluated for the trial configuration and the present configuration, respectively. Note that $\Delta S'$ comes solely from $S'_{\rm mixed}$ ($S'_{\rm com}$) when updating $V_\mu(z)$ with $\mu = 1,2$ ($\mu = 3,4$) thanks to the phase rotation of the auxiliary field (A.4). We repeat this procedure for all choices of μ and z.

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